New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2014

Due: 24 September 2014

Note: If any clarification is required, please contact Michael Albert (michael.albert@cs.otago.ac.nz).

1. Prove that for all positive real numbers a and b:
$$\frac{(a+b)^3}{4} \geq a^2b + ab^2.$$

2. Let ABC be a triangle in which the length of side AB is 4 units, and that of BC is 2 units. Let D be the point on AB at distance 3 units from A. Prove that the line perpendicular to AB through D, the angle bisector of $\angle ABC$, and the perpendicular bisector of BC all meet at a single point.

3. Find all pairs (x, y) of positive integers such that $(x+y)(x^2+9y)$ is the cube of a prime number.

4. Given 2014 points in the plane, no three of which are collinear, what is the minimum number of line segments that can be drawn connecting pairs of points in such a way that adding a single additional line segment of the same sort will always produce a triangle of three connected points?

5. Let ABC be an acute angled triangle. Let the altitude from C to AB meet AB at C' and have midpoint M, and let the altitude from B to AC meet AC at B' and have midpoint N. Let P be the point of intersection of AM and BB' and Q the point of intersection of AN and CC'. Prove that the point M, N, P and Q lie on a circle.

6. Determine all triples of positive integers a, b and c such that their least common multiple is equal to their sum.

7. Determine all pairs of real numbers (k, d) such that the system of equations:
$$x^3 + y^3 = 2$$
$$kx + d = y,$$
has no solutions (x, y) with x and y real numbers.

8. Michael wants to arrange a doubles tennis tournament among his friends. However, he has some peculiar conditions: the total number of matches should equal the total number of players, and every pair of friends should play as either teammates or opponents in at least one match. The number of players in a single match is four. What is the largest number of people who can take part in such a tournament?

9. Let AB be a line segment with midpoint I. A circle, centred at I has diameter less than the length of the segment. A triangle ABC is tangent to the circle on sides AC and BC. On AC a point X is given, and on BC a point Y is given such that XY is also tangent to the circle (in particular X lies between the point of tangency of the circle with AC and C, and similarly Y lies between the point of tangency of the circle with BC and C. Prove that $AX \cdot BY = AI \cdot BI$.

10. In the land of Microbablia the alphabet has only two letters, 'A' and 'B'. Not surprisingly, the inhabitants are obsessed with the band ABBA. Words in the local dialect with a high ABBA-factor are considered particularly lucky. To compute the ABBA-factor of a word you just count the number of occurrences of ABBA within the word (not necessarily consecutively). So for instance AABA has ABBA-factor 0, ABBA has ABBA-factor 1, AABBBA has ABBA-factor 6, and ABBABBA has ABBA factor 8. What is the greatest possible ABBA-factor for a 100 letter word?