New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2013

Due: 20th October 2013

1. You have a set of five weights, together with a balance that allows you to compare the weight of two things. The weights are known to be 10, 20, 30, 40 and 50 grams, but are otherwise identical except for their labels. The 10 and 50 gram weights are clearly labelled, but the labels have been erased on the remaining weights. Using the balance exactly once, is it possible to determine what one of the three unlabelled weights is? If so, explain how, and if not, explain why not.

2. Find all primes that can be written both as a sum and as a difference of two primes (note that 1 is *not* a prime).

3. Prove that for any positive integer $n > 2$ we can find n distinct positive integers, the sum of whose reciprocals is equal to 1.

4. Let C be a cube. By connecting the centres of the faces of C with lines we form an octahedron O. By connecting the centers of each face of O with lines we get a smaller cube C'. What is the ratio between the side length of C and the side length of C'?

5. Consider functions f from the whole numbers (non-negative integers) to the whole numbers that have the following properties:

   - For all x and y, $f(xy) = f(x)f(y)$,
   - $f(30) = 1$, and
   - for any n whose last digit is 7, $f(n) = 1$.

   Obviously, the function whose value at n is 1 for all n is one such function. Are there any others? If not, why not, and if so, what are they?

6. $ABCD$ is a quadrilateral having both an inscribed circle (one tangent to all four sides) with center I, and a circumscribed circle with center O. Let S be the point of intersection of the diagonals of $ABCD$. Show that if any two of S, I and O coincide, then $ABCD$ is a square (and hence all three coincide).

7. In a sequence of positive integers an *inversion* is a pair of positions such that the element in the position to the left is greater than the element in the position to the right. For instance the sequence 2,5,3,1,3 has five inversions – between the first and fourth positions, the second and all later positions, and between the third and fourth positions. What is the largest possible number of inversions in a sequence of positive integers whose sum is 2014?

8. Suppose that a and b are positive integers such that

$$c = a + \frac{b}{a} - \frac{1}{b}$$

is an integer. Prove that c is a perfect square.

9. Let ABC be a triangle with $\angle CAB > 45°$ and $\angle CBA > 45°$. Construct an isosceles right angled triangle RAB with AB as its hypotenuse and R inside ABC. Also construct isosceles right angled triangles ACQ and BCP having AC and BC respectively as their hypotenuses and lying entirely outside ABC. Show that $CQRP$ is a parallelogram.

10. Find the largest possible real number C such that for all pairs (x, y) of real numbers with $x \neq y$ and $xy = 2$,

$$\frac{\left((x+y)^2 - 6\right)\left((x-y)^2 + 8\right)}{(x-y)^2} \geq C.$$

Also determine for which pairs (x, y) equality holds.

11. Show that we cannot find 171 binary sequences (sequences of 0's and 1's), each of length 12 such that any two of them differ in at least four positions.

12. For a positive integer n, let $p(n)$ denote the largest prime divisor of n. Show that there exist infinitely many positive integers m such that $p(m-1) < p(m) < p(m+1)$.